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Research Paper

A Study on Symmetric Subgroups

A.Punitha Tharani [1] and V.Uma Maheswari [2]
[1]Associate Professor, [2]Research Scholar
[1,2] Department of Mathematics ,St.Mary's College (Autonomous) ,
Thoothukudi-628 001,Tamilnadu,India.
Affiliated to Manonmaniam Sundaranar University
Abishekapatti,Tirunelveli-627012,Tamilnadu,India.
Corresponding Author: V.Uma Maheswari

ABSTRACT: *In this note we define symmetric subgroups under the operators composition and plus circle compo.Also we derive some results based on the above concepts.*
KEYWORDS: *Symmetric groups,Symmetric subgroups , Composition ,Plus circle compo.*

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I. INTRODUCTION

In mathematics, the symmetric group on a set is the group consisting of all bijections of the set (all one-to-one and onto functions) from the set to itself with function composition as the group operation. The symmetric group is important to diverse areas of mathematics such as Galois theory, invariant theory, the representation theory of Lie groups, and combinatorics. Cayley's theorem states that every group G is isomorphic to a subgroup of the symmetric group on G.

II. PRELIMINARIES

Definition 2.1:
Let A be a finite set containing n elements. The set of all permutations of A is clearly a group under the composition of functions. This group is called the symmetric group of degree n and is denoted by S_n.
Definition 2.2:
Let G be a group, a subset H of G is called a subgroup of G if H itself is a group under the operation induced by G.
Definition 2.3: (Reverse Composition - O_R)
Let us consider a symmetric group S_2. The elements of S_2 are $\{\begin{pmatrix}1&2\\1&2\end{pmatrix},\begin{pmatrix}1&2\\2&1\end{pmatrix}\} = \{e,p_1\}$
The Reverse Composition is defined as in S_2, $e\, O_R\, p_1 = \begin{pmatrix}1&2\\1&2\end{pmatrix}\ O_R \begin{pmatrix}1&2\\2&1\end{pmatrix}$
The composition mapping is $1 \Rightarrow 1 \Rightarrow 2$ here we define the reverse composition mapping as
$1 \to 1 \to 2$ (i e) $2 \to 1$
similarly, $2 \to 2 \to 1$ (i e) $1 \to 2$

$$e\ O_R\ p_1 = \begin{pmatrix}1&2\\2&1\end{pmatrix} = p_1$$

and also $p_1\, O_R\, e = \begin{pmatrix}1&2\\1&2\end{pmatrix}\ O_R \begin{pmatrix}1&2\\2&1\end{pmatrix}$
(i e) $1 \to 1 \to 2 \Rightarrow 2 \to 1$
$2 \to 2 \to 1 \Rightarrow 1 \to 2.$

$$p_1\, O_R\, e = \begin{pmatrix}1&2\\1&2\end{pmatrix} = p_1$$

It's clearly O_R is also a binary operation.

Definition 2.4:
We define a new operation in composition mapping on S_3, that is called as plus circle compo,

Table 1.Plus circle compo

$+^o$	e	p_1	p_2	p_3	p_4	p_5
e	e	p_1	p_2	p_3	p_4	p_5
p_1	p_1	p_1	**e**	e	e	e
p_2	p_2	**e**	p_2	e	e	e
p_3	p_3	e	e	p_3	e	e
p_4	p_4	e	e	e	p_4	e
p_5	p_5	e	e	e	e	p_5

III. MAIN RESULTS

Definition 3.1:

Let S_2 be a set with a binary operation 'o' defined on it.Let $S \subseteq S_2$. If for each $e, p_1 \epsilon$ S, $e\ o\ p_1$ is in S. we say that S is Symmetric closed with respect to the binary operation'o'.In generally ,Let S_n be a set with a binary operation 'o' defined on it.Let $S \subseteq S_n$. If for each $e, p_1 \epsilon$ S, $e\ o\ p_1$ is in S. we say that S is Symmetric closed with respect to the binary operation'o'.

Example 3.2:

Let us consider a symmetric group S_2. The elements of S_2 are $\{\binom{1\ 2}{1\ 2},\binom{1\ 2}{2\ 1}\}$ ={e,p_1}

Let S={e,p_1}.Then $e\ o\ p_1 = p_1 \epsilon$ S ➔S is Symmetric closed in S_2.

Definition 3.3:

A subset S of a Symmetric group (S_2 ,o) is called a symmetric subgroup of S_2 if S forms a group with respect to the binary operation in S_2.

Definition 3.4:

A subset S of a Symmetric group (S_3 ,o) is called a symmetric subgroup of S_3 if S forms a group with respect to the binary operation in S_3.

Example 3.5:

Let us consider a symmetric group S_3. The elements of S_3

$\{\binom{1\,2\,3}{1\,2\,3},\binom{1\,2\,3}{2\,3\,1},\binom{1\,2\,3}{3\,1\,2},\binom{1\,2\,3}{1\,3\,2},\binom{1\,2\,3}{3\,2\,1},\binom{1\,2\,3}{2\,1\,3}\} = \{e, p_1, p_2, p_3, p_4, p_5\}$

Let $S=\{e,p_1, , p_2\}$.Then $e\ o\ p_1 = p_1 \epsilon$ S

(i)Clearly S is symmetric closed

(ii) Associative:

$(e\ o\ p_1)op_2 = p_1op_2 = e$ and $e\ o\ (p_1op_2) = e\ o(e) = e$

(iii) Identity:

There exists an element e ϵ S such that $e\ o\ p_1 = p_1\ o\ e = e$ *for all* $p_1 \epsilon$ S

(iv) Inverse:

For any element $p_1 \epsilon$ S, There exists an element $p_2 \epsilon$ S such that $p_2\ o\ p_1 = p_1\ o\ p_2 = e$

➔S is Symmetric closed in S_3.

Theorem 3.6:

Let S be symmetric subgroup of (S_2 ,o) then

(i) The identity element of S is the same as that of S_2.

(ii) For each $p_1 \epsilon$ S the inverse of p_1 in /s is the same as the inverse of p_1 in S_2.

Proof:

(i) Let e and e′ be two identity element of S_2.
Then $eo\ e' = e'$(since e is an identity element)
Also $e'oe = e$ (since e′ is an identity element)
$e = e'$

(ii) Let $P_1$and P_1' and P_1'' be two inverse of P,
$Here\ P_1oP_1 = P_1'oP_1 = e$ and $P_1oP_1'' = P_1''oP_1 = e$
$P_1' = P_1'o\ e = P_1'(P_1oP_1'') = (P_1'oP_1)P_1''$
$P_1' = e\ o\ P_1'' = P_1''$

The inverse of P_1 is unique

Theorem 3.7:
Let S be symmetric subgroup of (S_3 ,o) then
(i)The identity element of S is the same as that of S_3.
(ii) For each $p_1 \epsilon$ S the inverse of p_1 in /s is the same as the inverse of p_1 in S_3.
Proof:
(i) Let e and e′ be two identity element of S_3.
Then $eo\, e' = e'$(since e is an identity element)
Also $e'oe = e$ (since e′ is an identity element)
$e = e'$
(ii) Let $P_1$and P_1' and P_1'' be two inverse of P,
Here $P_1oP_1 = P_1'oP_1 = e$ *and*
$P_1oP_1'' = P_1''oP_1 = e$
$P_1' = P_1'o\, e = P_1'(P_1oP_1'') = (P_1'oP_1)P_1''$
$P_1' = e\, o\, P_1'' = P_1''$
The inverse of P_1 is unique
Theorem 3.8:
A Symmetric subgroup (S_2) can have at most one identity that are equal.
Proof:
Let the element of S_2= {e,p_1 }Take the symmetric subgroup,S ={e,p_1 }
If e and e′ are both identities.Then e = eoe′ = e′
Similarly, e′= e′oe = e =>e = e′
Therefore each symmetric subgroup have at most one identity, e = e′
Theorem 3.9:
A Symmetric subgroup (S_3) can have at most one identity that are equal.
Proof:
Let the element of S_3
$\{\binom{1\,2\,3}{1\,2\,3},\binom{1\,2\,3}{2\,3\,1},\binom{1\,2\,3}{3\,1\,2},\ \binom{1\,2\,3}{1\,3\,2},\ \binom{1\,2\,3}{3\,2\,1},\ \binom{1\,2\,3}{2\,1\,3}\} = \{e, p_1, p_2, p_3, p_4, p_5\}$
Take the symmetric subgroup,$S = \{e, p_1, p_2\}$
If e and e′ are both identities.Then e = eoe′ = e′Similarly, e′= e′oe = e=> e = e′
Therefore each symmetric subgroup have at most one identity, e = e′
Proposition 3.10:
Inverse of a symmetric subgroup (S_2 ,O) is also a symmetric subgroup.
Proof:
Let S_2 ={e,p_1} = $\{\binom{1\,2}{1\,2},\binom{1\,2}{2\,1}\}$
And let S={e,p_1}and S^{-1} = $\{e,p1\}^{-1}$ = $\{\binom{1\,2}{1\,2},\binom{1\,2}{2\,1}\}$
$S^{-1} = \{\binom{1\,2}{1\,2},\binom{1\,2}{2\,1}\}$($e^{-1} = \binom{1\,2}{1\,2}$$P_1^{-1} = \binom{1\,2}{2\,1}$)
S^{-1}= S is also a symmetric subgroup under composition
Proposition 3.11:
Inverse of a symmetric subgroup *(S_3,o)* is also a symmetric subgroup.
Proof:
Let $S_3 = \{e, p_1, p_2, p_3, p_4, p_5\}$
$S^{-1} = \{e, p_1, p_2\}^{-} = \{e^{-1}, p_1^{-1}, p_2^{-1}\}$
$e = \binom{1\,2\,3}{1\,2\,3}$, $p_1 = \binom{1\,2\,3}{2\,3\,1}$,$p_2 = \binom{1\,2\,3}{3\,1\,2}$
$(e)^{-1} = \binom{1\,2\,3}{1\,2\,3}^{-1} = \binom{1\,2\,3}{1\,2\,3} = e$
$(p_1)^{-1} = \binom{1\,2\,3}{2\,3\,1}^{-1} = \binom{1\,2\,3}{3\,1\,2} = p_2$
$(p_2)^{-1} = \binom{1\,2\,3}{3\,1\,2}^{-1} = \binom{1\,2\,3}{2\,3\,1} = p_1$
→ $S^{-1} = S = \{e, p_1, p_2\}$
→ Inverse of a symmetric subgroup *(S_3,o)* is also a symmetric subgroup.
Definition 3.12:
Let S_2 be a set with a binary operation 'O_R' defined on it. Let S⊆S_2. If for each e,$p_1 \epsilon$ S,
e O_R p_1 is in S. we say that S is Symmetric closed with respect to the binary operation'O_R'.
Example 3.13:
Let us consider a symmetric group S_2. The elements of S_2 are $\{\binom{1\,2}{1\,2},\binom{1\,2}{2\,1}\}$ ={e,p_1}
Let S={e,p_1}
Then e O_R $p_1 = p_1 \epsilon$ S
→S is Symmetric closed in S_2.

Definition 3.14:
Let S_3 be a set with a binary operation '$+^o$' defined on it.Let $S \subseteq S_3$. If for each $e, p_1 \epsilon$ S, $e +^o p_1$ is in S. we say that S is Symmetric closed with respect to the binary operation'$+^o$'.

Example 3.15:
Let us consider a symmetric group S_3. The elements of $S_3 \{\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix},\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix},\begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}, \begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}, \begin{pmatrix}1&2&3\\3&2&1\end{pmatrix}, \begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}\} = \{e, p_1, p_2, p_3, p_4, p_5\}$

Let $S=\{e, p_1, , p_2\}$

Then $e +^o p_1 = p_1 \epsilon$ S

(i)Clearly S is symmetric closed

(ii) Associative:

$(e+^o p_1) +^o p_2 = p_1 +^o p_2 = e$

$e +^o (p_1 +^o p_2) = e +^o (e) = e$

(iii) Identity:

There exists an element $e \epsilon$ S such that $e +^o p_1 = p_1 +^o e = e$ *for all* $p_1 \epsilon$ S

(iv) Inverse:

For any element $p_1 \epsilon$ S, There exists an element $p_2 \epsilon$ S such that $p_2 +^o p_1 = p_1 +^o p_2 = e$

➔S is Symmetric closed in S_3.

Definition 3.16:
A subset S of a Symmetric group (S_2 , O_R) is called a symmetric subgroup of S_2 if S forms a group with respect to the binary operation in S_2.

Definition 3.17:
A subset S of a Symmetric halfgroup (S_3 , $+^o$) is called a symmetric subgroup of S_3 if S forms a group with respect to the binary operation in S_3.

Theorem 3.18:
Let S be symmetric subgroup of (S_2 , O_R) then

(iii) The identity element of S is the same as that of S_2.

(iv) For each $p_1 \epsilon$ S the inverse of p_1 in /s is the same as the inverse of p_1 in S_2.

Proof:

(iii) Let e and e′ be two identity element of S_2.
Then $e \, O_R \, e' = e'$(since e is an identity element)
Also $e' \, O_R e = e$ (since e′ is an identity element)
$e = e'$

(iv) Let $P_1$and P_1' and P_1'' be two inverse of P,
Here $P_1 \, O_R \, P_1 = P_1' \, O_R \, P_1 = e$ *and*
$P_1 \, O_R \, P_1'' = P_1'' \, O_R \, P_1 = e$
$P_1' = P_1' \, O_R e = P_1' \, O_R \, (P_1 \, O_R \, P_1'')$
$= (P_1' \, O_R \, P_1) \, O_R \, P_1''$
$P_1' = e O_R \, P_1'' = P_1''$
The inverse of P_1 is unique

Theorem 3.19:
Let S be symmetric subgroup of (S_3 , $+^o$) then

(i)The identity element of S is the same as that of S_3.

(ii)For each $p_1 \epsilon$ S the inverse of p_1 in s is the same as the inverse of p_1 in S_3.

Proof:

(i)Let e and e′ be two identity element of S_3 .
Then $eo \, e' = e'$(since e is an identity element)
Also $e' +^o e = e$ (since e′ is an identity element)
$e = e'$

(ii)Let $P_1$and P_1' and P_1'' be two inverse of P,
Here $P_1 +^o P_1 = P_1' +^o P_1$
$= e$ *and*
$P_1 +^o P_1'' = P_1'' +^o P_1 = e$
$P_1' = P_1' +^o e = P_1' +^o (P_1 +^o P_1'')$
$= (P_1' +^o P_1) +^o P_1''$
$P_1' = e +^o P_1'' = P_1''$
The inverse of P_1 is unique

Theorem 3.20:
A Symmetric subgroup (S_2) can have at most one identity that are equal.

Proof:
Let the element of $S_2 = \{e, p_1\}$
Take the symmetric subgroup, $S = \{e, p_1\}$
If e and e′ are both identities.
Then $e = e\ O_R\ e' = e'$
Similarly, $e' = e'\ O_R\ e = e$
$e = e'$
Therefore each symmetric subgroup have at most one identity, e = e′

Theorem 3.21:
A Symmetric subgroup (S_3) can have at most one identity that are equal.

Proof:
Let the element of $S_3\{\binom{1\,2\,3}{1\,2\,3}, \binom{1\,2\,3}{2\,3\,1}, \binom{1\,2\,3}{3\,1\,2}, \binom{1\,2\,3}{1\,3\,2}, \binom{1\,2\,3}{3\,2\,1}, \binom{1\,2\,3}{2\,1\,3}\} = \{e, p_1, p_2, p_3, p_4, p_5\}$
Take the symmetric subgroup,
$$S = \{e, p_1, p_2\}$$
If e and e′ are both identities.
Then $e = e +^o e' = e'$
Similarly, $e' = e' +^o e = e$
$e = e'$
Therefore each symmetric subgroup have at most one identity, e = e′

Proposition 3.22:
Inverse of a symmetric subgroup (S_2 , O_R) is also a symmetric subgroup.

Proof:
Let $S_2 = \{e, p_1\} = \{\binom{1\,2}{1\,2}, \binom{1\,2}{2\,1}\}$
And let $S = \{e, p_1\}$ $S^{-1} = \{e, p1\}^{-1} = \{\binom{1\,2}{1\,2}, \binom{1\,2}{2\,1}\}$
$S^{-1} = \{\binom{1\,2}{1\,2}, \binom{1\,2}{2\,1}\}$($e^{-1} = \binom{1\,2}{1\,2}$ $P_1^{-1} = \binom{1\,2}{2\,1}$)
$S^{-1} = S$ is also a symmetric subgroup under composition

Proposition 3.23:
Inverse of a symmetric subgroup $(S_3, +^o)$ is also a symmetric subgroup.

Proof:
Let $S_3 = \{e, p_1, p_2, p_3, p_4, p_5\}$
$S^{-1} = \{e, p_1, p_2\}^{-1}$
$= \{e^{-1}, p_1^{-1}, p_2^{-1}\}$
$e = \binom{1\,2\,3}{1\,2\,3}$
$p_1 = \binom{1\,2\,3}{2\,3\,1}$
$p_2 = \binom{1\,2\,3}{3\,1\,2}$
$(e)^{-1} = \binom{1\,2\,3}{1\,2\,3}^{-1}$
$= \binom{1\,2\,3}{1\,2\,3} = e$
$(p_1)^{-1} = \binom{1\,2\,3}{2\,3\,1}^{-1}$
$= \binom{1\,2\,3}{3\,1\,2} = p_2$
$(p_2)^{-1} = \binom{1\,2\,3}{3\,1\,2}^{-1}$
$= \binom{1\,2\,3}{2\,3\,1} = p_1$
→ $S^{-1} = S = \{e, p_1, p_2\}$
→ Inverse of a symmetric subgroup $(S_3, +^o)$ is also a symmetric subgroup.

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**Corresponding Author: V.Uma Maheswari*